

On the regularized trace formula for a Sturm–Liouville boundary value problem with two delays and two potentials

NATAŠA PAVLOVIĆ KOMAZEC  0000-0002-3375-5667

ABSTRACT. This paper is devoted to a Sturm–Liouville boundary value problem with two constant delays $\tau_1, \tau_2 \in [\pi/3, \pi)$ and two potentials $q_1, q_2 \in L_2[0, \pi]$. We derive an asymptotic expansion of the characteristic function and obtain asymptotic formulas for the eigenvalues. Using the Lidskii–Sadovnichii contour integration method, we establish an explicit formula for the first regularized trace of the corresponding operator.

1. INTRODUCTION

Trace and regularized trace formulas for differential operators constitute an important part of spectral theory and have been studied extensively (see: [1, 9, 15]). For classical Sturm–Liouville operators, trace formulas establish relations between the spectral characteristics of an operator and the coefficients of the corresponding differential equation, together with the boundary conditions. Such formulas provide valuable information on the distribution of eigenvalues and play an important role in inverse spectral problems (see: [4]). In recent years considerable attention has been devoted to differential operators with retarded arguments (see: [2, 3, 14, 16–20, 22]). Such operators naturally arise in various applications: control theory, economics, population dynamics, and related fields. The presence of delays significantly complicates the spectral analysis. In particular, the structure of the characteristic function and the asymptotic behavior of the eigenvalues differ essentially from those in the classical Sturm–Liouville case. Two main approaches are commonly used for deriving regularized trace formulas: Levitan’s method and the contour integration method. Levitan’s method is based on Hadamard’s factorization theorem and has been applied in [5, 6, 10, 11]. The contour integration method is related to the argument principle and to the study

2020 *Mathematics Subject Classification.* Primary: 34B09; Secondary: 34B24, 34L20.

Key words and phrases. Differential operators with delays, Sturm–Liouville operator, Regularized trace formula.

Full paper. Received 13 May 2026, accepted 18 Jun 2026, available online 29 Jun 2026.

of remainders in complex analysis (see: [12, 21]). In [7], a general method for calculating regularized sums of zeros of entire functions of class K was developed. This method made it possible to obtain regularized trace formulas for a wide class of problems generated by ordinary differential equations on a finite interval. The application of this method, known as the Lidskii–Sadovnichii method, to differential equations with a delay was first presented in [13].

In this paper, we determine the first regularized trace of a Sturm–Liouville boundary value problem with two constant delays and two potentials by using the Lidskii–Sadovnichii method. Namely, we study the equation

$$(1) \quad -y''(x) + \sum_{k=1}^2 q_k(x)y(x - \tau_k) = \lambda y(x), \quad x \in [0, \pi],$$

together with the boundary conditions

$$(2) \quad y'(0) - hy(0) = 0,$$

and

$$(3) \quad y'(\pi) + Hy(\pi) = 0,$$

where $\frac{\pi}{3} \leq \tau_2 < \frac{\pi}{2} < 2\tau_2 \leq \tau_1 < \pi$, $h, H \in \mathbb{R} \setminus \{0\}$ and $\lambda = z^2$ is a spectral parametar. We assume that $q_1(x), q_2(x)$ are real-valued potential functions from $L_2[0, \pi]$, and such that $q_k(x) = 0$, $x \in [0, \tau_k]$, $k = 1, 2$.

We first derive an asymptotic formula for the corresponding characteristic function F_{τ_1, τ_2} . On this basis, we obtain asymptotic representation formulas for the eigenvalues and establish the first regularized trace formula for problem (1)–(3).

2. PRELIMINARIES

Lemma 1. *Let $y(x, z)$ be the solution of the differential equation*

$$(4) \quad -y''(x) + \sum_{k=1}^2 q_k(x)y(x - \tau_k) = z^2 y(x), \quad x \in [0, \pi],$$

which satisfies the initial conditions $y(0, z) = 1$, $y'(0, z) = h$ and for the functions $q_k(x)$, $k = 1, 2$, it holds $q_k(x) = 0$, $x \in [0, \tau_k]$. Then $y(x, z)$ satisfies the integral equation

$$(5) \quad y(x, z) = \cos zx + \frac{h}{z} \sin zx + \frac{1}{z} \sum_{k=1}^2 \int_{\tau_k}^x q_k(t) \sin z(x - t) y(t - \tau_k, z) dt.$$

Proof. Equation (4) can be written in the form

$$-y''(x, z) + z^2 y(x, z) = \sum_{k=1}^2 q_k(x)y(x - \tau_k, z).$$

General solution of the homogeneous equation $-y''(x, z) + z^2 y(x, z) = 0$ with initial conditions $y(0, z) = 1$, $y'(0, z) = h$ is

$$y_0(x, z) = \cos zx + \frac{h}{z} \sin zx.$$

By applying the method of variation of constants it is obtained

$$(6) \quad y(x, z) = \cos zx + \frac{h}{z} \sin zx + \frac{1}{z} \sum_{k=1}^2 \int_0^x q_k(t) \sin z(x-t) y(t - \tau_k, z) dt.$$

Based on the assumption, $q_k(t) = 0$ for $t < \tau_k$, equation (6) reduces to the integral equation (5), which should have been proven. $\square$

By the method of steps (see: [10, 13, 14]), it can be easily verified that the solution of the integral equation (5) on the interval $(\tau_1, \pi]$ is

$$(7) \quad \begin{aligned} y(x, z) = & \cos zx + \frac{h}{z} \sin zx + \frac{1}{z} \sum_{k=1}^2 b_{sc}^{(k)}(x, z) + \frac{h}{z^2} \sum_{k=1}^2 b_{s^2}^{(k)}(x, z) + \\ & + \frac{1}{z^2} b_{s^2c}^{(2)}(x, z) + \frac{h}{z^3} b_{s^3}^{(2)}(x, z), \end{aligned}$$

where

$$\begin{aligned} b_{sc}^{(k)}(x, z) &= \int_{\tau_k}^x q_k(t) \sin z(x-t) \cos z(t - \tau_k) dt, \\ b_{s^2}^{(k)}(x, z) &= \int_{\tau_k}^x q_k(t) \sin z(x-t) \sin z(t - \tau_k) dt, \\ b_{s^3}^{(2)}(x, z) &= \int_{2\tau_2}^x q_2(t) \sin z(x-t) b_{s^2}^{(2)}(t - \tau_2, z) dt, \\ b_{s^2c}^{(2)}(x, z) &= \int_{2\tau_2}^x q_2(t) \sin z(x-t) b_{sc}^{(2)}(t - \tau_2, z) dt. \end{aligned}$$

Let denote

$$F_{\tau_1, \tau_2}(\pi, z) = y'(\pi, z) + Hy(\pi, z).$$

For $x = \pi$ from (7) we get the characteristic function

$$\begin{aligned}
 (8) \quad F_{\tau_1, \tau_2}(z) = & \left(-z + \frac{hH}{z}\right) \sin \pi z + (h + H) \cos \pi z + \sum_{k=1}^2 b_{c^2}^{(k)}(z) + \\
 & + \frac{h}{z} \sum_{k=1}^2 b_{cs}^{(k)}(z) + \frac{H}{z} \sum_{k=1}^2 b_{sc}^{(k)}(z) + \frac{Hh}{z^2} \sum_{k=1}^2 b_{s^2}^{(k)}(z) + \\
 & + \frac{1}{z} b_{csc}^{(2)}(z) + \frac{h}{z^2} b_{cs^2}^{(2)}(z) + \frac{H}{z^2} b_{s^2c}^{(2)}(z) + \frac{Hh}{z^3} b_{s^3}^{(2)}(z),
 \end{aligned}$$

where

$$\begin{aligned}
 (9) \quad b_{cs}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t) \cos z(\pi - t) \sin z(t - \tau_k) dt, \\
 b_{c^2}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t) \cos z(\pi - t) \cos z(t - \tau_k) dt, \\
 b_{s^2c}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t) \sin z(\pi - t) b_{sc}^{(2)}(t - \tau_2, z) dt, \\
 b_{csc}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t) \cos z(\pi - t) b_{sc}^{(2)}(t - \tau_2, z) dt, \\
 b_{cs^2}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t) \cos z(\pi - t) b_{s^2}^{(2)}(t - \tau_2, z) dt.
 \end{aligned}$$

Here, to simplify the above-given equations for $F_{\tau_1, \tau_2}(z)$, we write (z) as the argument of the functions instead of (π, z) . Applying the trigonometric identities in expressions (9) leads to the asymptotic form of the characteristic function $F_{\tau_1, \tau_2}(z)$ given in expression (10) for $|z| \rightarrow \infty$

$$\begin{aligned}
 (10) \quad F_{\tau_1, \tau_2}(z) = & \left(-z + \frac{hH}{z}\right) \sin \pi z + (h + H) \cos \pi z + \frac{1}{2} \sum_{k=1}^2 \cos z(\pi - \tau_k) \int_{\tau_k}^{\pi} q_k(t) dt \\
 & + \frac{H + h}{2z} \sum_{k=1}^2 \sin z(\pi - \tau_k) \int_{\tau_k}^{\pi} q_k(t) dt + \frac{1}{2} \sum_{k=1}^2 \int_{\tau_k}^{\pi} q_k(t) \cos z(\pi - 2t + \tau_k) dt \\
 & + \frac{1}{4z} \sin z(\pi - 2\tau_2) \int_{2\tau_2}^{\pi} q_2(t) \int_{\tau_2}^{t-\tau_2} q_2(t_1) dt_1 dt + O\left(\frac{1}{|z|^2}\right).
 \end{aligned}$$

When we apply partial integration to integral $\int_{\tau_k}^{\pi} q_k(t) \cos z(\pi - 2t + \tau_k) dt$ and introducing notations

$$J_1^k = \int_{\tau_k}^{\pi} q_k(t) dt, \quad k = 1, 2; \quad J_2^{(2)} = \int_{2\tau_2}^{\pi} q_2(t) \int_{\tau_2}^{t-\tau_2} q_2(t_1) dt_1 dt,$$

the characteristic function given in (10) can be written in the following form

$$\begin{aligned} F_{\tau_1, \tau_2}(z) = & \left(-z + \frac{hH}{z} \right) \sin \pi z + (h + H) \cos \pi z \\ & + \frac{1}{2} \sum_{k=1}^2 J_1^{(k)} \cos z(\pi - \tau_k) + \frac{J_2^{(2)}}{4z} \sin z(\pi - 2\tau_2) \\ (11) \quad & + \frac{H + h}{2z} \sum_{k=1}^2 J_1^{(k)} \sin z(\pi - \tau_k) \\ & + \frac{1}{4z} \sum_{k=1}^2 \left(q_k(\pi) + q_k(\tau_k) \right) \sin z(\pi - \tau_k) + O\left(\frac{1}{|z|^2} \right). \end{aligned}$$

In order to be able to apply the Lidskii-Sadovnichii method, it is necessary to show that the function $F_{\tau_1, \tau_2}(z)$ is a function of class K .

Definition 1 ([7]). Let $f(z)$ be an entire function which for every integer h can be written in the form

$$f(z) = \sum_{k=0}^{N-1} e^{\alpha_k z} P_{k,h}(z),$$

where α_k are complex constants and

$$P_{k,h}(z) = z^{n_k} \sum_{\nu=0}^h \beta_{\nu}^{(k)} z^{-\nu} + o(z^{n_k-h}), \quad z \rightarrow \infty,$$

n_k are integers and $\beta_0^{(k)} \neq 0$, then the functions $f(z)$ are called functions of class K . The numbers α_k and $\beta_{\nu}^{(k)}$ are called parameters of the asymptotics of the function $f(z)$.

Lemma 2. *The characteristic function $F_{\tau_1, \tau_2}(z)$ of the considered Sturm–Liouville problem (1)–(3) belongs to the class K , and we can write it in the form*

$$\begin{aligned} (12) \quad F_{\tau_1, \tau_2}(z) = & \sum_{k=0}^2 e^{iz(\pi - k\tau_2)} P_k(z) + \sum_{k=1}^2 e^{-iz(\pi - k\tau_2)} P_{k+3}(z) + e^{iz(\pi - \tau_1)} P_3(z) \\ & + e^{-iz(\pi - \tau_1)} P_6(z) + e^{-i\pi z} P_7(z) + P_8(z), \quad |z| \rightarrow \infty, \end{aligned}$$

where

$$(13) \quad P_0(z) = -\frac{z}{2i} + \frac{hH}{2iz} + \frac{h+H}{2},$$

$$(14) \quad P_1(z) = \frac{1}{4}J_1^{(2)} + \frac{1}{2i} \left[\frac{H+h}{2z}J_1^{(2)} + \frac{1}{4z}(q_2(\pi) + q_2(\tau_2)) \right],$$

$$(15) \quad P_2(z) = \frac{J_2^{(2)}}{8iz}, \quad P_3(z) = \frac{1}{4}J_1^{(1)} + \frac{1}{2i} \left[\frac{H+h}{2z}J_1^{(1)} + \frac{1}{4z}(q_1(\pi) + q_1(\tau_1)) \right],$$

$$(16) \quad P_4(z) = \frac{1}{4}J_1^{(2)} - \frac{1}{2i} \left[\frac{H+h}{2z}J_1^{(2)} + \frac{1}{4z}(q_2(\pi) + q_2(\tau_2)) \right],$$

$$(17) \quad P_5(z) = -\frac{J_2^{(2)}}{8iz},$$

$$(18) \quad P_6(z) = \frac{1}{4}J_1^{(1)} - \frac{1}{2i} \left[\frac{H+h}{2z}J_1^{(1)} + \frac{1}{4z}(q_1(\pi) + q_1(\tau_1)) \right],$$

$$(19) \quad P_7(z) = \frac{z}{2i} - \frac{hH}{2iz} + \frac{h+H}{2},$$

$$(20) \quad P_8(z) = 0.$$

Proof. As is known, trigonometric functions can be written in exponential form

$$\begin{aligned} \sin \pi z &= \frac{e^{i\pi z} - e^{-i\pi z}}{2i}, & \cos \pi z &= \frac{e^{i\pi z} + e^{-i\pi z}}{2}, \\ \cos z(\pi - \tau_k) &= \frac{e^{iz(\pi - \tau_k)} + e^{-iz(\pi - \tau_k)}}{2}, \\ \sin z(\pi - k\tau_k) &= \frac{e^{iz(\pi - k\tau_k)} - e^{-iz(\pi - k\tau_k)}}{2i}, \quad k = 1, 2. \end{aligned}$$

Substituting these identities into the asymptotic formula of the characteristic function given in (11), we obtain

$$\begin{aligned} F_{\tau_1, \tau_2}(z) &= \left[-\frac{z}{2i} + \frac{hH}{2iz} + \frac{h+H}{2} \right] e^{i\pi z} + \left[\frac{z}{2i} - \frac{hH}{2iz} + \frac{h+H}{2} \right] e^{-i\pi z} \\ &\quad + \frac{J_2^{(2)}}{8iz} e^{iz(\pi - 2\tau_2)} - \frac{J_2^{(2)}}{8iz} e^{-iz(\pi - 2\tau_2)} + \frac{1}{4} \sum_{k=1}^2 J_1^{(k)} e^{iz(\pi - \tau_k)} \\ (21) \quad &\quad + \frac{1}{2i} \sum_{k=1}^2 \left[\frac{H+h}{2z} J_1^{(k)} + \frac{1}{4z} (q_k(\pi) + q_k(\tau_k)) \right] e^{iz(\pi - \tau_k)} \\ &\quad - \frac{1}{2i} \sum_{k=1}^2 \left[\frac{H+h}{2z} J_1^{(k)} + \frac{1}{4z} (q_k(\pi) + q_k(\tau_k)) \right] e^{-iz(\pi - \tau_k)} \\ &\quad + \frac{1}{4} \sum_{k=1}^2 J_1^{(k)} e^{-iz(\pi - \tau_k)} + O\left(\frac{1}{|z|^2}\right), \quad |z| \rightarrow \infty. \end{aligned}$$

When we insert the expressions given by (13)–(20) into equation (21), we obtain that the characteristic function can be written in the form (12).

Thus, based on Definition 1, we see that the function $F_{\tau_1, \tau_2}(z)$ belongs to the class K . $\square$

3. MAIN RESULTS

In this section, we derive asymptotic formulas for the eigenvalues and establish the regularized trace formula for problem (1)–(3). It is known, see [9], that the spectrum of problem (1)–(3) is discrete and that $z_n \sim n + o(1)$, $n \rightarrow \infty$. Since the principal objective of this paper is to compute the first regularized trace of the Sturm–Liouville operator associated with problem (1)–(3), it is enough to obtain an asymptotic expansion of the zeros z_n of the corresponding characteristic function in the form

$$z_n(\tau_k) = n + \frac{C_1(n, \tau_k)}{n} + \frac{C_2(n, \tau_k)}{n^2} + o\left(\frac{1}{n^2}\right), \quad k = 1, 2; \quad n \rightarrow \infty.$$

Using the analogous procedure shown in papers [10] and [11], we obtain that the zeros of the function $F_{\tau_1, \tau_2}(z)$ have an asymptotic shape

$$(22) \quad \begin{aligned} z_n(\tau_k) = n + & \frac{c_{01} + \sum_{k=1}^2 c_{k1} \cos n\tau_k}{n} + \frac{\sum_{k=1}^2 c_{k2} \sin n\tau_k}{n^2} \\ & + \frac{c_{42} \sin n(\tau_1 + \tau_2) + c_{52} \sin n(\tau_1 - \tau_2)}{n^2} \\ & + \frac{c_{32} \sin 2n\tau_1 + c_{62} \sin 2n\tau_2}{n^2} + o\left(\frac{1}{n^2}\right), \end{aligned}$$

where

$$\begin{aligned} c_{01} &= \frac{h + H}{\pi}; \\ c_{k1} &= \frac{J_1^{(k)}}{2\pi}, \quad k = 1, 2; \\ c_{k2} &= \frac{1}{2\pi^2} \left[-\tau_k J_1^{(k)}(h + H) - \frac{\pi}{2}(q_k(\pi) + q_k(\tau_k)) \right], \quad k = 1, 2; \\ c_{32} &= \frac{1}{8\pi^2} (J_1^{(1)})^2 (\pi - \tau_1); \\ c_{42} &= \frac{1}{8\pi^2} J_1^{(1)} J_1^{(2)} (2\pi - (\tau_1 + \tau_2)); \\ c_{52} &= -\frac{1}{8\pi^2} J_1^{(1)} J_1^{(2)} (\tau_1 - \tau_2); \\ c_{62} &= \frac{(J_1^{(2)})^2 (\pi - \tau_2) - 2\pi J_2^{(2)}}{8\pi^2}. \end{aligned}$$

Since the eigenvalues λ_n are given by $\lambda_n = z_n^2$, where z_n are the zeros of the characteristic function, it follows from (22) that

$$(23) \quad \begin{aligned} \lambda_n(\tau_1, \tau_2) = & n^2 + 2c_{01} + 2 \sum_{k=1}^2 c_{k1} \cos n\tau_k + \frac{2}{n} \sum_{k=1}^2 c_{k2} \sin n\tau_k \\ & + \frac{2}{n} [c_{42} \sin n(\tau_1 + \tau_2) + c_{52} \sin n(\tau_1 - \tau_2)] \\ & + \frac{2}{n} (c_{32} \sin 2n\tau_1 + c_{62} \sin 2n\tau_2) + o\left(\frac{1}{n}\right), \quad n \rightarrow \infty. \end{aligned}$$

The first regularized trace is defined as the sum obtained after subtracting from each eigenvalue the leading asymptotic terms responsible for the divergence of the usual eigenvalue series.

Theorem 1. *Let $\lambda_n(\tau_1, \tau_2)$, given in expression (23), denote the eigenvalues of problem (1)–(3). Then, the first regularized trace is given by*

$$(24) \quad S_1 = \sum_{n=1}^{\infty} \left\{ \lambda_n(\tau_1, \tau_2) - n^2 - 2c_{01} - 2 \sum_{k=1}^2 c_{k1} \cos n\tau_k \right\} = \omega_3 - \Phi_2$$

where

$$\omega_3 = \pi(h + H)^3 + h^2 + H^2, \quad \Phi_2 = -\frac{h + H}{\pi} - \frac{1}{2\pi} \sum_{k=1}^2 J_1^{(k)}.$$

Proof. The coefficients ω_ν can be determined by means of the following formula (see: [7, 8, 13])

$$(25) \quad \frac{F'(z, \tau)}{F(z, \tau)} \sim \sum_{\nu=0}^{\infty} \frac{\omega_\nu(\tau)}{z^\nu}, \quad z \rightarrow \infty.$$

Substituting $z = -iy$ into the asymptotic relation (12) and letting $y \rightarrow \infty$, we obtain

$$F_{\tau_1, \tau_2}(-iy) = \frac{e^{\pi y}}{2} y \left[1 + \frac{h + H}{y} + \frac{hH}{y^2} + O(e^{-\tau_2 y}) \right].$$

It follows from (25) that

$$(26) \quad \frac{F'_{\tau_1, \tau_2}(-iy)}{F_{\tau_1, \tau_2}(-iy)} = \frac{\pi + \frac{\pi(h+H)+1}{y} + \frac{\pi hH}{y^2} - \frac{Hh}{y^3} + O(e^{-\tau_2 y})}{1 + \frac{h+H}{y} + \frac{hH}{y^2} + O(e^{-\tau_2 y})}.$$

Therefore, relation (26) implies

$$(27) \quad \frac{F'_{\tau_1, \tau_2}(-iy)}{F_{\tau_1, \tau_2}(-iy)} = \pi + \frac{1}{y} - \frac{h + H}{y^2} + \frac{\pi(h + H)^3 + h^2 + H^2}{y^3} + O(e^{-\tau_2 y}).$$

Using (27), we obtain the following coefficients:

$$\omega_0 = \pi, \quad \omega_1 = 1, \quad \omega_2 = -(h + H), \quad \omega_3 = \pi(h + H)^3 + h^2 + H^2.$$

The quantity Φ_2 can be evaluated by means of the following formula ([13])

$$\Phi_2(-2, \tau_1, \tau_2) = \zeta(-2) + 2c_{01}\zeta(0) + 2c_{11}\zeta_{\cos}(0, \tau_1) + 2c_{21}\zeta_{\cos}(0, \tau_2),$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \zeta_{\cos}(z, \tau) = \sum_{n=1}^{\infty} \frac{\cos n\tau}{n^z}.$$

Accordingly

$$\zeta(-2) = 0, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta_{\cos}(0, \tau_k) = -\frac{1}{2}, \quad k = 1, 2.$$

Substituting the previously obtained values into the formula for Φ_2 , we obtain

$$\Phi_2(-2, \tau_1, \tau_2) = -\frac{h+H}{\pi} - \frac{1}{2\pi} \sum_{k=1}^2 J_1^{(k)}.$$

Consequently, the first regularized trace for problem (1)–(3) is given by

$$S_1 = \pi(h+H)^3 + h^2 + H^2 + \frac{h+H}{\pi} + \frac{1}{2\pi} \sum_{k=1}^2 J_1^{(k)}.$$

This completes the proof. $\square$

4. CONCLUSION

In this paper, we studied a Sturm–Liouville boundary value problem with two delayed arguments and two potential functions and derived its regularized trace formula by using the Lidskii–Sadovnichii method. The main contribution of the paper is the derivation of a regularized trace formula in which the influence of the boundary parameters, the delayed arguments, and the potential functions is explicitly represented. The obtained result generalizes the classical trace formulas for Sturm–Liouville operators and shows how delay terms modify the spectral structure of the problem. The proposed method provides a useful framework for further studies of spectral problems with delayed arguments, including higher regularized traces and inverse spectral problems for more general classes of differential operators.

REFERENCES

- [1] L.A. Dikii, *Trace formulas for Sturm–Liouville differential operators*, Uspekhi Matematicheskikh Nauk, XIII (3) (81) (1958), 111–143.
- [2] N. Djurić, B. Vojvodić, *Inverse problem for Dirac operators with a constant delay less than half the length of the interval*, Applicable Analysis and Discrete Mathematics, 17 (1) (2023), 249–261.

- [3] N. Djurić, S. Buterin, *On an open question in recovering Sturm-Liouville-type operators with delay*, Applied Mathematics Letters, 113 (2021), Article ID: 106862, 1-6.
- [4] G. Freiling, V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publishers, New York, Huntington, 2008.
- [5] F. Hira, N. Altınışık, *A trace formula for discontinuous eigenvalue problem*, Filomat, 31 (8) (2017), 2425-2431.
- [6] B.M. Levitan, *Calculation of the regularized trace for the Sturm-Liouville operator*, Uspekhi Matematicheskikh Nauk, XIX (1) (115) (1964), 161-165.
- [7] V.B. Lidskii, V.A. Sadovnichii, *Regularized sums of zeros of a class of entire functions*, Functional Analysis and its Applications, 1 (2) (1967), 133-139.
- [8] V.B. Lidskii, V.A. Sadovnichii, *Asymptotic formulas for zeros of a class of entire functions*, Mathematics of the USSR-Sbornik, 4 (4) (1968), 519-527.
- [9] S.B. Norkin, *Second Order Differential Equations with a Delay Argument*, Nauka, Moscow, 1965.
- [10] N. Pavlović, M. Pikula, B. Vojvodić, *First regularized trace of the limit assignment of Sturm-Liouville type with two constant delays*, Filomat, 29 (1) (2015), 51-62.
- [11] N. Pavlović Komazec, B. Vojvodić, *The First Regularized trace of the Sturm-Liouville operator with Robin boudary conditions*, Sarajevo Journals of Mathematics, 20 (1) (2024), 173-188.
- [12] M. Pikula, *Regularized Traces of Higher-Order Differential Operators with Retarded Argument*, Differentsial'nye Uravneniya, 21 (6) (1985), 986-991.
- [13] M. Pikula, *Regularized traces of a differential operator of Sturm-Liouville type with retarded argument*, Differentsial'nye Uravneniya, 26 (1) (1990), 103-109.
- [14] M. Pikula, V. Vladičić, B. Vojvodić, *Inverse spectral problems for Sturm-Liouville operators with a constant delay less than half the length of the interval and Robin boundary conditions*, Results in Mathematics, 74 (2019), Article ID: 45, 1-13.
- [15] V.A. Sadovnichii, V.E. Podolskii, *Traces of operators*, Russian Mathematical Surveys, 61 (5) (2006), 885-953.
- [16] V. Vladičić, M. Bošković, B. Vojvodić, *Inverse Problems for Sturm-Liouville-Type Differential Equation with a Constant Delay Under Dirichlet/Polynomial Boundary Conditions*, Bulletin of the Iranian Mathematical Society, 48 (4) (2022), 1829-1843.
- [17] B. Vojvodić, M. Pikula, V. Vladičić, *Inverse problems for Sturm-Liouville differential operators with two constant delays under Robin boundary conditions*, Results in Applied Mathematics, 5 (2020), Article ID: 100082, 1-7.
- [18] B. Vojvodić, N. Pavlović Komazec, *Inverse problems for Sturm-Liouville operator with potential functions from $L_2[0, \pi]$* , Mathematica Montisnigri, 49 (2020), 28-38.
- [19] B. Vojvodić, N. Pavlović Komazec, F.A. Çetinkaya, *Recovering differential operators with two retarded arguments*, Boletín de la Sociedad Matemática Mexicana, 28 (2022), Article ID: 68, 1-21.
- [20] B. Vojvodić, N. Djurić, V. Vladičić, *On recovering dirac operators with two delays*, Complex Analysis and Operator Theory, 18 (2024), Article ID: 105, 1-21.

-
- [21] C.F. Yang, *Trace and inverse problem of discontinuous Sturm–Liouville operator with retarded argument*, Journal of Mathematical Analysis and Applications, 395 (2012) 30-41.
- [22] F. Wang, C.-F. Yang, S. Buterin, N. Djurić, *Inverse spectral problems for Dirac-type operators with global delay on a star graph*, Analysis and Mathematical Physics, 14 (2024), Article ID: 24, 1-16.

NATAŠA PAVLOVIĆ KOMAZEC

UNIVERSITY OF EAST SARAJEVO

FACULTY OF ELECTRICAL ENGINEERING

VUKA KARADŽIĆA 30

71123 EAST SARAJEVO

BOSNIA AND HERZEGOVINA

E-mail address: `natasa.pavlovic@etf.ues.rs.ba`